Moduli of boundary polarized CY surface pairs /$\mathbb{C}$

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builds on previous work w/ Ascher, Bejleri, DeVleming, Inchiostro, Liu, Wang ABBDILW

§0. Compact Moduli of varieties

A. canonically pol varieties

KSBA $\quad M^{KSBA} = \{ \text{proj var } X \text{ w/ slc sing's} + K_X \text{ ample} \}$ $\qquad$ slc $\approx$ lc + nodes in codim 1

B. Fano varieties

K-moduli $\quad M^{Kss} := \{ \text{proj K-ss Fano varieties} \} / \sim_S$

K-ss $\Rightarrow$ kH

Key features

- boundedness when fix dim + vol
- proj moduli spaces w/ ample CM lb
- works for pairs (X,D)

Goal construct moduli of CYs interpolates between K + KSBA

§ 1. Boundary polarized CY pairs

<u>Def</u> a <u>bp CY pair</u> (X,D) is a proj variety X w/ $\mathbb{Q}$-divisor D st

(X,D) slc, $K_X + D \sim_{\mathbb{Q}} 0$, D is ample.

<u>Examples</u>

<u>Src</u> (minimal lc center on dlt modification)

I. $(\mathbb{P}^2, \frac{1}{2}\text{ sextic})$ — $(\mathbb{P}^2, \frac{1}{2}(\text{sextic}))$

2:1 cover of pair is a deg 2 K3

II. a) E elliptic curve L ample lb

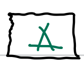
$(X = C_p(E,L),\ D = \text{divisor at } \infty)$

E

b) $(\mathbb{P}^2,\ Q_1 + \frac{1}{2}Q_2)$

$(\mathbb{P}^1, \frac{1}{4}(p_1 + \cdots + p_4))$

III. $(\mathbb{P}^2, \{xyz=0\})$



pt

<u>Type</u>: I	II	III
klt	$\dim \mathrm{Src} = 1$	$\dim \mathrm{Src} = 0$

§2. Previous approach to moduli

If (X,D) is a klt bpCY pair, then

- $(X, (1+\varepsilon)D)$ KSBA stable can pol pair
- $(X, (1-\varepsilon)D)$ K-ss log Fano pair

for $0 < \varepsilon \ll 1$.

So get compact moduli space for perturbed pair

	$\underline{M^{KSBA}}$	$\underline{M^{K}}$
general theory	KX, Birr	ADL, Zho
$(\mathbb{P}^2, \frac{3}{d} C_d)$ $4 \leq d \leq 6$	Hac, AET	ADL
$(X, \frac{1}{9}(L_1 + \cdots + L_{27}))$ $X \subset \mathbb{P}^3$ smooth cubic	HKT	Zhao

$\underline{M^{KSBA}}$ $\underline{M^{K}}$

- Y K3 surface w/ non-symp aut

  σ of deg d.

  $\mathrm{Fix}\,\sigma$ = curve of genus $g \geq 2$

$\overline{Y} = \mathrm{Proj}\ R(Y, \mathrm{Fix}\,\sigma)$

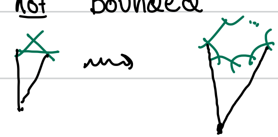
$X = \overline{Y}/\overline{\sigma}$ $D = \frac{d-1}{d} R$

AEH, AE

<u>Goal</u> construct M^{CY} interpolates M^{K} and M^{KSBA}

Q) What should M^{CY} parametrize? <u>Naive Answer</u> All bp CY pairs $/\sim$

<u>Issue</u> bp CY w/ fixed dim + vol <u>not</u> bounded

- $(\mathbb{P}^2, \{xyz=0\}) \rightsquigarrow$ 
- $(\mathbb{P}(a^2, b^2, c^2), \{xyz=0\})$ $a^2+b^2+c^2 = 3abc$

<u>Def</u> two bpCY pairs are <u>S-equivalent</u> if they admit isotriv degens to a common bp CY pair

$$\begin{tikzcd}[column sep=small]
(X, D) \arrow[dr, squiggly] & & (X', D') \arrow[dl, squiggly] \\
& (X_0, D_0) &
\end{tikzcd}$$

<u>ex</u> previous pairs are S-equivalent

<u>Prop</u> S-equivalence preserves Type + Src   numerical invariants

<u>Prop</u> (ABBDILW) Exists a <u>locally</u> finite type alg stack param bpCY pairs (X, D) w/ fixed numerical invariants.

§ 3. Main Results

Setup: $(\mathcal{X}, \mathcal{D}) \longrightarrow T$ family of bp CY $\overbrace{\text{surface}}^{\dim \mathcal{X} - \dim T = 2}$ pairs over finite type scheme T

$$\mathcal{M}^{CY} := \overline{\text{image} \left(T \longrightarrow \text{moduli stack of bp CY pairs} \right)}^{\text{sn}}$$

Rem

- concretely $\mathcal{M}^{CY}(\mathbb{C})$ = bp CY pairs that are limits of fibers of $(\mathcal{X}, \mathcal{D}) \to T$
- $\mathcal{M}^{CY}$ is not finite type in general
- open substacks

open loci where $(X, (1 \pm \varepsilon)D)$ is K-ss / KSBA stable

$$\begin{array}{ccccc} \mathcal{M}^{K} & \subset & \mathcal{M}^{CY} & \supset & \mathcal{M}^{KSBA} \\ \downarrow & & & & \downarrow \\ M^{K} & & & & M^{KSBA} \end{array}$$

admit proj
good moduli spaces

Thm (B-Liu)

Exists surj morphism to a proj scheme $\mathcal{M}^{CY} \longrightarrow M^{CY}$ st

① $M^{CY}(\mathbb{C}) = \mathcal{M}^{CY}(\mathbb{C}) / \sim_S$

② Exist wall crossing maps

$$M^K \longrightarrow M^{CY} \longleftarrow M^{KSBA}$$

③ Hodge line bundle is ample on M^{CY}

Note • Proved in ABBDILW when $\mathcal{X}_t \simeq \mathbb{P}^2$ for $t \in T$ general

• Hodge lb $f : (X, D) \longrightarrow S$ family of bp CYs

$$\lambda_{Hodge, f} = f_* \mathcal{O}_X (r(K_X + D))^{\otimes 1/r} \qquad r>0 \text{ sufficiently div}$$

$\approx$ moduli divisor in MMP

• compactification is anologue of BB

__ex__ ($\mathbb{P}^2$, elliptic curve) $\quad M^{CY} \simeq \mathbb{P}^1_j$

$j \neq \infty$  $\sim_S$ 

$j = \infty$ unbounded S-equiv class

__ex__ ($\mathbb{P}^2$, $\frac{1}{2}$ sextic) $\quad M^{CY} \simeq F_2^{BB}$

__Thm__ For moduli of K3 surf w/ non-symplectic automorphism, $(M^{CY})^n$ = BB comp of period domain

§4. Construction of M^{CY}

① bounded substacks

For $m \geq 1$, $\mathcal{M}_m$ = substack of $\mathcal{M}^{CY}$ param pairs (X, D) in $\mathcal{M}^{CY}(\mathbb{C})$

w/ $\mathrm{ind}_x(K_X) \leq m \quad \forall x \in X$

Kollár, Fujino $\Rightarrow$ $\mathcal{M}_m$ finite type

② $\exists$ morphism to good moduli space $\mathcal{M}_m \to M_m$ for $m \gg 0$

By AHLH, need to show $\mathcal{M}_m$ S-comple + Θ-reductive

$$\begin{tikzcd}
S \arrow[r] \arrow[dr, dashed] & \mathcal{M}^{CY} \\
S \setminus 0 \arrow[u, hook, bend left] \arrow[r] & \mathcal{M}_m \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

uses:
- $\mathcal{M}^{CY}$ is S-complete + Θ-red (ABBDILW)
- deformation theory of slc surfaces (KSB)

Gorenstein + slt $\Rightarrow$ hypersurface sing inclusions $\mathcal{M}_m \subset \mathcal{M}_{m+1} \subset \mathcal{M}_{m+2} \subset \cdots$

induce: $M_m \longrightarrow M_{m+1} \longrightarrow M_{m+2} \to \cdots$

3. stabilization: for $m \gg 0$, $M_m \to M_{m+1}$ is $\simeq$

analyze loci:

Type I + II: loci in $\mathcal{M}^{CY}$ is bounded

Type III: loci in M_m is discrete for $m \gg 0$

Show (X, D) type III bpCY surface pair $\Rightarrow$ admits isotriv degen to pair (X_0, D_0) whose normalization toric

Set: $M^{CY} := M_m$ for $m \gg 0$

4. L_{Hodge} on M^{CY} is ample

Results on moduli divisor by Kaw, Amb, Kol $\Rightarrow$ L_{Hodge} is ample assuming Src of points $p \in M^{CY}$ have maximal variation

Type I: trivially holds

Type II: need to show for each Src pair of dim 1 there are finitely many S-eqvl classes

Type III: holds since locus is discrete